Skew shape representations are irreducible

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Abstract. In this paper all of the classical constructions of A. Young are generalized to affine Hecke algebras of type A. It is proved that the calibrated irreducible representations of the affine Hecke algebra are indexed by placed skew shapes and that these representations can be constructed explicitly with a generalization of Young's seminormal construction of the irreducible representations of the symmetric group. The seminormal construction of an irreducible calibrated module does not produce a basis on which the affine Hecke algebra acts integrally but using it one is able to pick out a different basis, an analogue of Young's natural basis, which does generate an integral lattice in the module. Analogues of the "Garnir relations" play an important role in the proof. The Littlewood-Richardson coefficients arise naturally as the decomposition multiplicities for the restriction of an irreducible representation of the affine Hecke algebra to the Iwahori-Hecke algebra.

0. Introduction

My recent work [Ra3], [Ra4] on the representations of affine Hecke algebras has been strongly motivated by the classical theory of Young tableaux. This research has resulted in the generalization of many of A. Young's constructions to general finite root systems. With these generalizations of standard Young tableaux one is able to use Young's classical "seminormal construction" to construct irreducible representations of affine Hecke algebras corresponding to arbitrary finite crystallographic root systems.

Because the classical combinatorics of Young tableaux is much more advanced than that of the newly developed generalization it is often possible to give simpler proofs and more extensive results for the case of affine Hecke algebras of type A. The purpose of this paper is to compile some of these results and proofs. In particular, we obtain a generalization of Young's natural basis and derive certain induction and restriction rules which are not yet available in the general case. It will also be more clear from the exposition here, how the generalization of the Young tableau theory given in [Ra3] and [Ra4] relates to the classical setup, something which is not always obvious when working in the general root system context.

The main results

(1) *The definition of calibrated representations, and the classification and construction of all irreducible calibrated representations of the affine Hecke algebra type A.*

These representations are indexed by placed skew shapes. The dimension of an irreducible calibrated representation is the number of standard tableaux of the corresponding skew shape and

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the representation is constructed by explicit formulas which give the action of each generator of the affine Hecke algebra on a specific basis, the elements of which are indexed by standard Young tableaux. This is a generalization of the constructions of A. Young [Y], P. Hoefsmit [Ho], H. Wenzl [Wz], and Ariki and Koike [AK] (see [Ra2] for a review of some of the unpublished results of Hoefsmit). Parts of Theorem 4.1 were first discovered by Cherednik and are stated (without proof) in [Ch]. I am grateful to A. Zelevinsky for pointing this out to me and to I. Cherednik for some informative discussions.

(2) *The definition of an analogue of Young's natural basis for each irreducible calibrated representation.*

Young's natural basis is the one that is most often used in the study of irreducible representations of the symmetric group, it is the one that is usually taken as the basis of the "Specht module", see for example [JK], [Sg], [Fu]. It has the wonderful property that it is an *integral* basis for the module, i.e. the matrices representating the action of the symmetric group on this basis contain integer entries. This is especially important because it opens the door to a combinatorial study of the modular representations of the symmetric group.

Using the analogue of the seminormal basis for the irreducible calibrated representations of the affine Hecke algebra we can define an analogue of Young's natural basis in each of these representations. As desired, this basis is an integral basis of the module; the matrices representing the action of the affine Hecke algebra on this basis have all entries in the ring $\mathbb{Z}[q, q^{-1}]$. These results are a q-analogue of some of the results in [GW].

One of the pleasant surprises one has when generalizing Young's natural basis from this point of view is that the "Garnir relations" take a particularly simple form: If $\{v_L\}$ is Young's seminormal basis and $\{n_L\}$ is Young's natural basis then the relations

$$v_L = 0, \qquad \text{when } L \text{ is not a standard tableau,}$$

are the Garnir relations. One recovers the Garnir relations in their classical form by expanding the v_L in terms of the n_L.

(3) *The classical Littlewood-Richardson coefficients describe the decomposition of the restriction of an irreducible representation of the affine Hecke algebra to the Iwahori-Hecke algebra.*

This result gives a completely new (and unexpected) representation theoretic interpretation of the Littlewood-Richardson coefficients.

(4) *Skew shapes arise naturally as indexes for the irreducible calibrated representations of the affine Hecke algebra of type A.*

Until now skew shapes have appeared in the combinatorial literature as something of a novelty, a useful combinatorial tool which indexes some strangely well-behaved representations of the symmetric group. It has always been a surprise that the combinatorics of the irreducible representations of the symmetric group generalizes so beautifully to this special class of highly reducible representations of the symmetric group.

This fact is no longer strange. In fact, these representations are *irreducible* representations of the affine Hecke algebra, and thus are basic and fundamental. Several of the skew Schur function identities in [Mac] I can be given representation theoretic interpretations in this context, see Theorem 6.2 and Corollary 6.3.

Acknowledgements

This paper is part of a series [Ra3-5] [RR1-2] of papers on representations of affine Hecke algebras. During this work I have benefited from conversations with many people. To choose only a few, there were discussions with S. Fomin, F. Knop, L. Solomon, M. Vazirani and N. Wallach

which played an important role in my progress. There were several times when I tapped into J. Stembridge's fountain of useful knowledge about root systems. G. Benkart was a very patient listener on many occasions. H. Barcelo, P. Deligne, T. Halverson, R. Macpherson and R. Simion all gave large amounts of time to let me tell them my story and every one of these sessions was helpful to me in solidifying my understanding.

I single out Jacqui Ramagge with special thanks for everything she has done to help with this project: from the most mundane typing and picture drawing to deep intense mathematical conversations which helped to sort out many pieces of this theory. Her immense contribution is evident in that some of the papers in this series on representations of affine Hecke algebras are joint papers.

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1. The affine Hecke algebra of type A

Affine braids. There are three common ways of depicting affine braids [Cr], [GL], [Jo]:

(a) As braids in a (slightly thickened) cylinder,

(b) As braids in a (slightly thickened) annulus,

(c) As braids with a flagpole.

See Figure 1. The multiplication is by placing one cylinder on top of another, placing one annulus inside another, or placing one flagpole braid on top of another. These are equivalent formulations: an annulus can be made into a cylinder by turning up the edges, and a cylindrical braid can be made into a flagpole braid by putting a flagpole down the middle of the cylinder and pushing the pole over to the left so that the strings begin and end to its right.

The group formed by the affine braids with n strands is the *affine braid group* $\tilde{\mathcal{B}}_n$ of type A. Let ω, T_i for $0 \le i \le n-1$, and x_i for $1 \le i \le n$, be as given in Figure 2. The following identities can be checked by drawing pictures:

$$
\begin{array}{lll}
\text{(a)}\ \ T_iT_j = T_jT_i, & \text{for } |i-j|>1, & \\
\text{(b)}\ \ T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, & \text{for } 0 \le i \le n-1, & \\
\text{(c)}\ \ \omega T_i \omega^{-1} = T_{i-1}, & \text{for } 0 \le i \le n-1, & \\
\text{(d)}\ \ x_iT_j = T_jx_i, & \text{if } |i-j|>1, & \\
\text{(e)}\ \ x_{i+1} = T_ix_iT_i, & \text{for } 1 \le i \le n-1, & (1.1)\\
\text{(f)}\ \ x_ix_j = x_jx_i, & \text{for } 1 \le i,j, \le n, & \\
\text{(g)}\ \ x_nx_1^{-1} = T_0T_{n-1}\cdots T_2T_1T_2\cdots T_{n-1}, & & \\
\text{(h)}\ \ x_n = \omega T_1T_2\cdots T_{n-1}, & & \\
\text{(i)}\ \ \omega^n = x_1x_2\cdots x_n, & &
\end{array}
$$

where the indices on the elements T_i are taken modulo n. The elements T_i, $0 \le i \le n-1$, and ω generate $\tilde{\mathcal{B}}_n$. The *braid group* is the subgroup $\mathcal{B}_n$ generated by the T_i, $1 \le i \le n-1$. The elements x_i, $1 \le i \le n$, generate an abelian group $X \subseteq \tilde{\mathcal{B}}_n$. If $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \mathbb{Z}^n$ define

$$x^\gamma = x_1^{\gamma_1} x_2^{\gamma_2} \cdots x_n^{\gamma_n}. \tag{1.2}$$

The symmetric group S_n acts on $\mathbb{Z}^n$ by permuting the coordinates. This action induces an action on X by

$$wx^\gamma = x^{w\gamma}, \qquad \text{for } w \in S_n,\ \gamma \in \mathbb{Z}^n.$$

The affine Hecke algebra. Fix an element $q \in \mathbb{C}^*$ which is not a root of unity. The *affine Hecke algebra* $\tilde{H}_n$ is the quotient of the group algebra $\mathbb{C}\tilde{\mathcal{B}}_n$ by the relations

$$T_i^2 = (q - q^{-1})T_i + 1, \qquad 0 \le i \le n. \tag{1.3}$$

The images of T_i, x_i and ω in $\tilde{H}_n$ are again denoted by T_i, x_i and ω. The Laurent polynomial ring $\mathbb{C}[X] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is a (large) commutative subalgebra of $\tilde{H}_n$.

The relations $T_i^{-1} = T_i - (q - q^{-1})$ and $x_{i+1} = T_i x_i T_i$ can be used to derive the identities

$$x_{i+1}T_i = T_i x_i + (q - q^{-1})x_{i+1}, \qquad \text{and} \qquad x_i T_i = T_i x_{i+1} - (q - q^{-1})x_{i+1}. \tag{1.4}$$

More generally, if $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \mathbb{Z}^n$ then

$$x^\gamma T_i = T_i x^{s_i\gamma} + (q - q^{-1})\frac{(x^\gamma - x^{s_i\gamma})x_{i+1}}{x_{i+1} - x_i}, \tag{1.5}$$

where $s_i \in S_n$ is the simple transposition $(i, i+1)$. The right hand term in this expression can always be written as a Laurent polynomial in $x_1, \ldots, x_n$. This important relation is due to Bernstein, Zelevinsky and Lusztig [Lu]. The affine Hecke algebra $\tilde{H}_n$ can be defined as the algebra generated by T_i, $1 \le i \le n$, and x_i, $1 \le i \le n$ subject to the relations in (1.1a), (1.1b), (1.1f), (1.3) and (1.5).

The symmetric group. The *simple transpositions* are the elements $s_i = (i, i+1)$, $1 \le i \le n-1$, in S_n. A *reduced word* for a permutation $w \in S_n$ is an expression $w = s_{i_1} \cdots s_{i_p}$ of minimal length. This minimal length is called the *length* $\ell(w)$ of w. The symmetric group S_n is partially ordered by the Bruhat-Chevalley order: $v \le w$ if a reduced expression $s_{i_1} \cdots s_{i_p}$ for w has a subword $s_{i_{k_1}} \cdots s_{i_{k_\ell}}$, $1 \le k_1 < \cdots < k_\ell \le p$ which is equal to v in S_n.

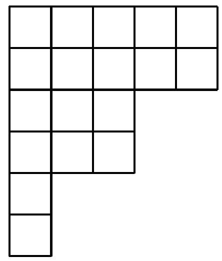
The Iwahori-Hecke algebra. The *Iwahori-Hecke algebra* H_n is the subalgebra of $\tilde{H}_n$ generated by the elements T_i, $1 \le i \le n-1$. For each $w \in S_n$ let

$$T_w = T_{i_1} \cdots T_{i_p}, \tag{1.6}$$

where $w = s_{i_1} \cdots s_{i_p}$ is a reduced word for w. Since the T_i satisfy the braid relations (1.1a,b), the element T_w is independent of the choice of the reduced word of w. The elements T_w, $w \in S_n$, are a basis of H_n [Bou, IV §2 Ex. 23].

2. Tableau combinatorics

Skew shapes and standard tableaux. A partition λ is a collection of n boxes in a corner. We shall conform to the conventions in [Mac] and assume that gravity goes up and to the left.



Any partition λ can be identified with the sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$ where λ_i is the number of boxes in row i of λ. The rows and columns are numbered in the same way as for matrices. In the example above we have $\lambda = (553311)$. If λ and μ are partitions such that $\mu_i \leq \lambda_i$ for all i we write $\mu \subseteq \lambda$. The *skew shape* λ/μ consists of all boxes of λ which are not in μ. Any skew shape is a union of connected components. Number the boxes of each skew shape λ/μ along major diagonals from southwest to northeast and

$$\text{write } \mathrm{box}_i \text{ to indicate the box numbered } i.$$

Let λ/μ be a skew shape with n boxes. A *standard tableau of shape* λ/μ is a filling of the boxes in the skew shape λ/μ with the numbers $1, \ldots, n$ such that the numbers increase from left to right in each row and from top to bottom down each column. Let

$$\mathcal{F}^{\lambda/\mu} = \{\text{standard tableaux of shape } \lambda/\mu\}.$$

The *column reading tableau* C of shape λ/μ is the standard tableau obtained by entering the numbers $1, 2, \ldots, n$ consecutively down the columns of λ/μ, beginning with the southwest most connected component and filling the columns from left to right. The *row reading tableau* R of shape λ/μ is the standard tableau obtained by entering the numbers $1, 2, \ldots, n$ left to right across the rows of λ/μ, beginning with the northeast most connected component and filling the rows from top to bottom. In general, if L is a standard tableau and $w \in S_n$ then wL will denote the filling of λ/μ obtained by permuting the entries of L according to the permutation w.

Proposition 2.1. [BW, Theorem 7.1] *Given a standard tableau L of shape λ/μ define the word of L to be permutation*

$$w_L = \begin{pmatrix} 1 & \cdots & n \\ L(\mathrm{box}_1) & \ldots & L(\mathrm{box}_n) \end{pmatrix}$$

where $L(\mathrm{box}_i)$ is the entry in box_i of L. Let C and R be the column reading and row reading tableaux of shape λ/μ, respectively. The map

$$\begin{array}{ccc} \mathcal{F}^{\lambda/\mu} & \longrightarrow & S_n \\ L & \longmapsto & w_L \end{array}$$

defines a bijection from $\mathcal{F}^{\lambda/\mu}$ to the interval $[w_C, w_R]$ in S_n (in the Bruhat-Chevalley order).

Placed skew shapes. Let $\mathbb{R} + i[0, 2\pi/\ln(q^2)) = \{a + bi \mid a \in \mathbb{R}, 0 \le b \le 2\pi/\ln(q^2)\} \subseteq \mathbb{C}$. If q is a positive real number then the function

$$\begin{array}{ccc} \mathbb{R} + i[0, 2\pi/\ln(q^2)) & \longrightarrow & \mathbb{C}^* \\ x & \longmapsto & q^{2x} = e^{\ln(q^2)x} \end{array}$$

is a bijection. The elements of $[0, 1) + i[0, 2\pi/\ln(q^2))$ index the $\mathbb{Z}$-cosets in $\mathbb{R} + i[0, 2\pi/\ln(q^2))$.

A *placed skew shape* is a pair $(c, \lambda/\mu)$ consisting of a skew shape λ/μ and a *content function*

$$c\colon \{\text{boxes of } \lambda/\mu\} \longrightarrow \mathbb{R} + i[0, 2\pi/\ln(q^2)) \qquad \text{such that}$$

$$\begin{array}{lc} c(\mathrm{box}_j) \ge c(\mathrm{box}_i), & \text{if } i < j \text{ and } c(\mathrm{box}_j) - c(\mathrm{box}_i) \in \mathbb{Z}, \\ c(\mathrm{box}_j) = c(\mathrm{box}_i) + 1, & \text{if and only if } \mathrm{box}_i \text{ and } \mathrm{box}_j \text{ are on adjacent diagonals, and} \\ c(\mathrm{box}_i) = c(\mathrm{box}_j), & \text{if and only if } \mathrm{box}_i \text{ and } \mathrm{box}_j \text{ are on the same diagonal.} \end{array}$$

This is a generalization of the usual notion of the content of a box in a partition (see [Mac] I §1 Ex. 3).

Suppose that $(c, \lambda/\mu)$ is a placed skew shape such that c takes values in $\mathbb{Z}$. One can visualize $(c, \lambda/\mu)$ by placing λ/μ on a piece of infinite graph paper where the diagonals of the graph paper are indexed consecutively (with elements of $\mathbb{Z}$) from southeast to northwest. The *content* of a box b is the index $c(b)$ of the diagonal that b is on. In the general case, when c takes values in $\mathbb{R} + i[0, 2\pi/\ln(q^2))$, one imagines a book with r pages of infinite graph paper where the diagonals of the graph paper are indexed consecutively (with elements of $\mathbb{Z}$) from southeast to northwest. The pages are numbered by values $\beta_1, \ldots, \beta_r$ from the set $[0, 1) + i[0, 2\pi/\ln(q^2))$ and there is a skew shape $\lambda^{(k)}/\mu^{(k)}$ placed on page β_k. The skew shape λ/μ is a union of the disjoint skew shapes $\lambda^{(i)}/\mu^{(i)}$,

$$\lambda/\mu = \lambda^{(1)}/\mu^{(1)} \cup \cdots \cup \lambda^{(r)}/\mu^{(r)},$$

and the content function is given by

$$c(b) = (\text{page number of the page containing } b) + (\text{index of the diagonal containing } b).$$

Example. The following diagrams illustrate standard tableaux and the numbering of boxes in a skew shape λ/μ.

$$\begin{array}{ccccccccc} & & & & & 10 & 12 & 13 & 14 \\ & & & & 6 & 8 & 11 & & \\ & & & & 5 & 7 & 9 & & \\ & & & 4 & & & & & \\ 2 & 3 & & & & & & & \\ 1 & & & & & & & & \end{array}$$

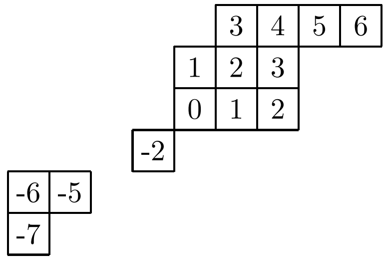
λ/μ with boxes numbered

$$\begin{array}{ccccccccc} & & & & & 3 & 4 & 9 & 12 \\ & & & & 1 & 5 & 10 & & \\ & & & & 7 & 13 & 14 & & \\ & & & 2 & & & & & \\ 6 & 8 & & & & & & & \\ 11 & & & & & & & & \end{array}$$

A standard tableau L of shape λ/μ

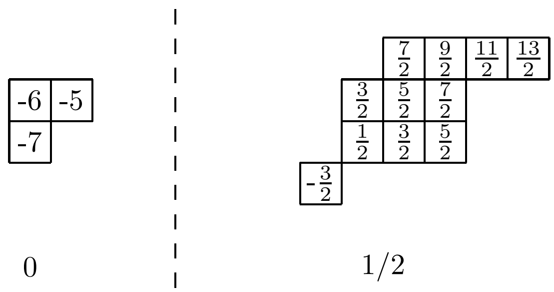
The word of the standard tableau L is the permutation $w_L = (11, 6, 8, 2, 7, 1, 13, 5, 14, 3, 10, 4, 9, 12)$ (in one-line notation).

The following picture shows the contents of the boxes in the placed skew shape $(c, \lambda/\mu)$ such that the sequence $(c(\mathrm{box}_1), \ldots, c(\mathrm{box}_n))$ is $(-7, -6, -5, -2, 0, 1, 1, 2, 2, 3, 3, 4, 5, 6)$.



Contents of the boxes of $(c, \lambda/\mu)$

The following picture shows the contents of the boxes in the placed skew shape $(c', \lambda/\mu)$ such that $(c(\mathrm{box}_1), \ldots, c(\mathrm{box}_n)) = (-7, -6, -5, -3/2, 1/2, 3/2, 3/2, 5/2, 5/2, 7/2, 7/2, 9/2, 11/2, 13/2)$.



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Lemma 2.2. *Let $(c, \lambda/\mu)$ be a placed skew shape with n boxes and let L be a standard tableau of shape λ/μ. Let $L(i)$ denote the box containing i in L. The content sequence*

$$(c(L(1)), \ldots, c(L(n))$$

uniquely determines the shape $(c, \lambda/\mu)$ and the standard tableau L.

Proof. Proceed by induction on the number of boxes of L. If L has only one box then the content sequence $(c(L(1))$ determines the placement of that box. Assume that L has n boxes. Let L' be the standard tableau determined by removing the box containing n from L. Then L' is also of skew shape and the content sequence of L' is $(c(L(1)), \ldots, c(L(n-1)))$. By the induction hypothesis we can reconstruct L' from its content sequence. Then $c(L(n))$ determines the diagonal which must contain box n in L. So L' and $c(L(n))$ determine L uniquely. ∎

3. Weights and weight spaces

A finite dimensional $\tilde{H}_n$-module is *calibrated* if it has a basis of simultaneous eigenvectors for the x_i, $1 \le i \le n$. In other words, M is calibrated if it has a basis $\{v_t\}$ such that for all v_t in the basis and all $1 \le i \le n$,

$$x_i v_t = t_i v_t, \qquad \text{for some } t_i \in \mathbb{C}^*.$$

Weights. Let X be the abelian group generated by the elements $x_1, \ldots, x_n \in \tilde{H}_n$ and let

$$T = \{\text{group homomorphisms } X \to \mathbb{C}^*\}.$$

The *torus* T can be identified with $(\mathbb{C}^*)^n$ by identifying the element $t = (t_1, \ldots, t_n) \in (\mathbb{C}^*)^n$ with the homomorphism given by $t(x_i) = t_i$, $1 \le i \le n$. The symmetric group S_n acts on $\mathbb{Z}^n$ by permuting coordinates and this action induces an action of S_n on T given by

$$(wt)(x^\gamma) = x^{w^{-1}\gamma}, \qquad \text{for } w \in S_n,\ \gamma \in \mathbb{Z}^n,$$

with notation as in (1.2).

Weight spaces. Let M be a finite dimensional $\tilde{H}_n$-module. For each $t = (t_1, \ldots, t_n) \in T$ the *t-weight space of M* and the *generalized t-weight space* are the subspaces

$$M_t = \{m \in M \mid x_i m = t_i m \text{ for all } 1 \le i \le n\} \qquad \text{and}$$

$$M_t^{\mathrm{gen}} = \{m \in M \mid \text{for each } 1 \le i \le n,\ (x_i - t_i)^k m = 0 \text{ for some } k \in \mathbb{Z}_{>0}\},$$

respectively. From the definitions, $M_t \subseteq M_t^{\mathrm{gen}}$ and M is calibrated if and only if $M_t^{\mathrm{gen}} = M_t$ for all $t \in T$. If $M_t^{\mathrm{gen}} \ne 0$ then $M_t \ne 0$. In general $M \ne \bigoplus_{t \in T} M_t$, but we do have

$$M = \bigoplus_{t \in T} M_t^{\mathrm{gen}}.$$

This is a decomposition of M into Jordan blocks for the action of $\mathbb{C}[X] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. The set of *weights of M* is the set

$$\mathrm{supp}(M) = \{t \in T \mid M_t^{\mathrm{gen}} \ne 0\}. \tag{3.1}$$

An element of M_t is called a *weight vector* of *weight t*.

The τ operators. The maps $\tau_i \colon M_t^{\mathrm{gen}} \to M_{s_i t}^{\mathrm{gen}}$ defined below are local operators on M in the sense that they act on each generalized weight space M_t^{gen} of M separately. The operator τ_i is only defined on the generalized weight spaces M_t^{gen} such that $t_i \ne t_{i+1}$.

Proposition 3.2. *Let $t = (t_1, \ldots, t_n) \in T$ be such that $t_i \ne t_{i+1}$ and let M be a finite dimensional $\tilde{H}_n$-module. Define*

$$\begin{array}{rccc} \tau_i\colon & M_t^{\mathrm{gen}} & \longrightarrow & M_{s_i t}^{\mathrm{gen}} \\ & m & \longmapsto & \left(T_i - \dfrac{(q-q^{-1})x_{i+1}}{x_{i+1} - x_i}\right) m. \end{array}$$

(a) *The map $\tau_i \colon M_t^{\mathrm{gen}} \longrightarrow M_{s_i t}^{\mathrm{gen}}$ is well defined.*

(b) *As operators on M_t^{gen}*

$$\begin{aligned} x_i \tau_i &= \tau_i x_{i+1}, \qquad x_{i+1}\tau_i = \tau_i x_i\,, \qquad \text{and} \qquad x_j \tau_i = \tau_i x_j, && \text{if } j \ne i, i+1, \\ \tau_i \tau_i &= \frac{(q x_{i+1} - q^{-1} x_i)(q x_i - q^{-1} x_{i+1})}{(x_{i+1} - x_i)(x_i - x_{i+1})}, && \text{if } 1 \le i \le n-1, \\ \tau_i \tau_j &= \tau_j \tau_i, && \text{if } |i-j| > 1, \\ \tau_i \tau_{i+1} \tau_i &= \tau_{i+1} \tau_i \tau_{i+1}, && 1 \le i \le n-1, \end{aligned}$$

whenever both sides are well defined.

Proof. (a) Note that $(q-q^{-1})x_{i+1}/(x_{i+1}-x_i)$ is not a well defined element of $\tilde{H}_n$ or $\mathbb{C}[x_1^{\pm 1},\ldots,x_n^{\pm 1}]$ since it is a power series and not a Laurent polynomial. Because of this we will be careful to view $(q-q^{-1})x_{i+1}/(x_{i+1}-x_i)$ only as an operator on M_t^{gen}. Let us describe this operator more precisely.

The element $x_i x_{i+1}^{-1}$ acts on M_t^{gen} by $t_i t_{i+1}^{-1}$ times a unipotent transformation. As an operator on M_t^{gen}, $(1-x_i x_{i+1}^{-1}) = x_{i+1}/(x_{i+1}-x_i)$ is invertible since it has determinant $(1-t_i t_{i+1}^{-1})^d$ where $d = \dim(M_t^{\mathrm{gen}})$. Since this determinant is nonzero $(q-q^{-1})x_{i+1}/(x_{i+1}-x_i) = (q-q^{-1})(1-x_i x_{i+1}^{-1})^{-1}$ is a well defined operator on M_t^{gen}. Thus the definition of τ_i makes sense.

The operator identities $x_i\tau_i = \tau_i x_{i+1}$ $x_{i+1}\tau_i = \tau_i x_i$, and $x_j\tau_j = \tau_i x_j$, if $i \neq i, i+1$, now follow easily from the definition of the τ_i and the identities in (1.4). These identities imply that τ_i maps M_t^{gen} into $M_{s_i t}^{\mathrm{gen}}$.

All of the operator identities in part (b) are proved by straightforward calculations of the same flavour as the calculation of $\tau_i\tau_i$ given below. We shall not give the details for the other cases. The only one which is really tedious is the calculation for the proof of $\tau_{i+1}\tau_i\tau_{i+1} = \tau_i\tau_{i+1}\tau_i$. For a more pleasant (but less elementary) proof of this identity see Proposition 2.7 in [Ra3].

Since $t_i \neq t_{i+1}$ both $\tau_i\colon M_t^{\mathrm{gen}} \to M_{s_i t}^{\mathrm{gen}}$ and $\tau_i\colon M_{s_i t}^{\mathrm{gen}} \to M_t^{\mathrm{gen}}$ are well defined. Let $m \in M_t^{\mathrm{gen}}$. Then

$$
\begin{aligned}
\tau_i\tau_i m &= \left(T_i - \frac{(q-q^{-1})x_{i+1}}{x_{i+1}-x_i}\right)\left(T_i - \frac{(q-q^{-1})x_{i+1}}{x_{i+1}-x_i}\right) m \\
&= \left(T_i^2 - T_i\frac{(q-q^{-1})x_{i+1}}{x_{i+1}-x_i} - \frac{(q-q^{-1})x_{i+1}}{x_{i+1}-x_i}T_i + \frac{(q-q^{-1})^2 x_{i+1}^2}{(x_{i+1}-x_i)^2}\right) m \\
&= \Bigg((q-q^{-1})T_i + 1 - T_i\frac{(q-q^{-1})x_{i+1}}{x_{i+1}-x_i} - T_i\frac{(q-q^{-1})x_i}{x_i-x_{i+1}} \\
&\qquad -(q-q^{-1})^2\frac{x_{i+1}}{x_{i+1}-x_i}\left(\frac{x_{i+1}}{x_{i+1}-x_i} - \frac{x_i}{x_i-x_{i+1}}\right) + \frac{(q-q^{-1})^2 x_{i+1}^2}{(x_{i+1}-x_i)^2}\Bigg) m \\
&= \left((q-q^{-1})T_i + 1 - (q-q^{-1})T_i + (q-q^{-1})^2\frac{x_i x_{i+1}}{(x_{i+1}-x_i)(x_i-x_{i+1})}\right) m \\
&= \frac{q^2 x_i x_{i+1} - 2x_i x_{i+1} + q^{-2}x_i x_{i+1} - x_i^2 + 2x_i x_{i+1} - x_{i+1}^2}{(x_{i+1}-x_i)(x_i-x_{i+1})} m \\
&= \frac{(qx_{i+1} - q^{-1}x_i)(qx_i - q^{-1}x_{i+1})}{(x_{i+1}-x_i)(x_i-x_{i+1})} m. \qquad \blacksquare
\end{aligned}
$$

Let $w \in S_n$. Let $w = s_{i_1}\cdots s_{i_p}$ be a reduced word for w and define

$$\tau_w = \tau_{i_1}\cdots\tau_{i_p}. \tag{3.3}$$

Since the τ-operators satisfy the braid relations the operator τ_w is independent of the choice of the reduced word for w. The operator τ_w is a well defined operator on M_t^{gen} if $t = (t_1,\ldots,t_n)$ is such that $t_i \neq t_j$ for all pairs $i < j$ such that $w(i) > w(j)$. One may use the relations in (1.5) to rewrite τ_w in the form

$$\tau_w = \sum_{u \le w} T_w a_{uw}(x_1,\ldots,x_n)$$

where $a_{uw}(x_1,\ldots,x_n)$ are rational functions in the variables $x_1,\ldots,x_n$. (The functions $a_{uw}(x_1,\ldots,x_n)$ are analogues of the Harish-Chandra *c-function*, see [Mac2, 4.1] and [Op, Theorem 5.3].) If

$t = (t_1, \ldots, t_n)$ is such that $t_i \neq t_j$ for all pairs $i < j$ such that $w(i) > w(j)$ then the expression

$$\tau_w\Big|_t = \sum_{u \le w} T_w a_{uw}(t_1, \ldots, t_n) \tag{3.4}$$

is a well defined element of the Iwahori-Hecke algebra H_n. If $w = uv$ with $\ell(w) = \ell(u) + \ell(v)$ then

$$\tau_w\Big|_t = \tau_u\Big|_{vt} \tau_v\Big|_t. \tag{3.5}$$

The following result will be crucial to the proof of Theorem 5.5. This result is due to D. Barbasch and P. Diaconis [D] (in the $q = 1$ case). The proof given below is a q-version of a proof for the $q = 1$ case given by S. Fomin [Fo].

Proposition 3.6. *Let w_0 be the longest element of S_n,*

$$w_0 = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix}.$$

Let $a \in \mathbb{C}^*$ and fix $t = (a, aq^2, aq^4, \ldots, aq^{2(n-1)})$. Then

$$\tau_{w_0}\Big|_t = \sum_{w \in S_n} T_w (-q)^{\ell(w_0) - \ell(w)},$$

where $\ell(w_0) = \binom{n}{2}$.

Proof. Let $1 \le k \le n$. Then there is a $v \in S_n$ such that $w_0 = v s_k$ and $\ell(w_0) = \ell(v) + 1$. So

$$\tau_{w_0} = \tau_v \tau_k = \tau_v \left(T_k - \frac{(q - q^{-1}) x_{k+1}}{x_{k+1} - x_k} \right)$$

and

$$\tau_{w_0}\Big|_t = \tau_v\Big|_{s_k t} \left(T_k - \frac{(q - q^{-1}) t_{k+1}}{t_{k+1} - t_k} \right) = \tau_v\Big|_{s_k t} \left(T_k - \frac{(q - q^{-1}) q^2 t_k}{(q^2 - 1) t_k} \right) = \tau_v\Big|_{s_k t} (T_k - q).$$

Right multiplying by $T_k + q^{-1}$ and using the relation (1.3) gives

$$\tau_{w_0}\Big|_t (T_k + q^{-1}) = \tau_v\Big|_{s_k t} (T_k - q)(T_k + q^{-1}) = 0.$$

The element $h = \sum_{w \in S_n} T_w (-q)^{\ell(w_0) - \ell(w)}$ is a multiple of the minimal central idempotent in H_n corresponding to the representation ϕ given by $\phi(T_k) = -q^{-1}$, for all $1 \le k \le n$. Up to multiplication by constants, it is the unique element in H_n such that $h(T_k + q^{-1}) = 0$ for all $1 \le k \le n$. The lemma follows by noting that the coefficients of T_{w_0} in h and $\tau_{w_0}\Big|_t$ are both 1. ∎

The action of the τ-operators on weight vectors will be particularly important to the proofs of the results in later sections. Let us record the following facts.

Let M be an $\tilde{H}_n$-module and let m_t be a weight vector in M of weight $t = (t_1, \ldots, t_n)$.

(3.7a) If $t_i \neq t_{i+1}$ then

$$\tau_i m_t = \left(T_i - \frac{(q - q^{-1}) x_{i+1}}{x_i - x_{i+1}} \right) m_t = \left(T_i - \frac{(q - q^{-1}) t_{i+1}}{t_i - t_{i+1}} \right) m_t$$

is a weight vector of weight $s_i t$.

(3.7b) By the second set of identities in Proposition 3.2 (b), $\tau_i \tau_i m_t = (q t_{i+1} - q^{-1} t_i)(q t_i - q^{-1} t_{i+1})(t_{i+1} - t_i)^{-1}(t_i - t_{i+1})^{-1} m_t$. Thus

$$\text{If } t_i \neq t_{i+1} \text{ and } t_i \neq q^{\pm 2} t_{i+1} \text{ then } \tau_i m_t \neq 0.$$

4. Classification and construction of calibrated representations

The following theorem classifies and constructs all irreducible calibrated representations of the affine Hecke algebra $\tilde{H}_n$. It shows that the theory of standard Young tableaux plays an intrinsic role in the combinatorics of the representations of the affine Hecke algebra. The construction given in Theorem 4.1 is a direct generalization of A. Young's classical "seminormal construction" of the irreducible representations of the symmetric group [Y]. Young's construction was generalized to Iwahori-Hecke algebras of type A by Hoefsmit [Ho] and Wenzl [Wz] independently, to Iwahori-Hecke algebras of types B and D by Hoefsmit [Ho] and to cyclotomic Hecke algebras by Ariki and Koike [AK]. It can be shown that all of these previous generalizations are special cases of the construction for affine Hecke algebras given here. Recently, this construction has been generalized even further [Ra3], to affine Hecke algebras of arbitrary Lie type. Some parts of Theorem 4.1 are due, originally, to I. Cherednik, and are stated in [Ch, §3].

Garsia and Wachs [GW] showed that the theory of standard Young tableaux and Young's constructions play an important role in the combinatorics of the skew representations of the symmetric group. At that time it was not known that these representations are actually *irreducible* as representations of the affine Hecke algebra!!

Theorem 4.1. *Let $(c, \lambda/\mu)$ be a placed skew shape with n boxes. Define an action of $\tilde{H}_n$ on the vector space*

$$\tilde{H}^{(c,\lambda/\mu)} = \mathbb{C}\text{-span}\{v_L \mid L \text{ is a standard tableau of shape } \lambda/\mu\}$$

by the formulas

$$x_i v_L = q^{2c(L(i))} v_L,$$
$$T_i v_L = (T_i)_{LL} v_L + (q^{-1} + (T_i)_{LL}) v_{s_i L},$$

where $s_i L$ is the same as L except that the entries i and $i+1$ are interchanged,

$$(T_i)_{LL} = \frac{q - q^{-1}}{1 - q^{2(c(L(i)) - c(L(i+1)))}}, \qquad v_{s_i L} = 0 \text{ if } s_i L \text{ is not a standard tableau,}$$

and $L(i)$ denotes the box of L containing the entry i.

(a) $\tilde{H}^{(c,\lambda/\mu)}$ is a calibrated irreducible $\tilde{H}_n$-module.
(b) The modules $\tilde{H}^{(c,\lambda/\mu)}$ are non-isomorphic.
(c) Every irreducible calibrated $\tilde{H}_n$-module is isomorphic to $\tilde{H}^{(c,\lambda/\mu)}$ for some placed skew shape $(c, \lambda/\mu)$.

STEP 1. The given formulas for the action of $\tilde{H}^{(c,\lambda/\mu)}$ define an $\tilde{H}_n$-module.

Proof. If L is a standard tableau then the entries i and $i+1$ cannot appear in the same diagonal in L. Thus, for all standard tableaux L, $c(L(i)) \neq c(L(i+1))$ and for this reason the constant $(T_i)_{LL}$ is always well defined.

Let L be a standard tableau of shape λ/μ. Then $(T_i)_{LL} + (T_i)_{s_iL,s_iL} = q - q^{-1}$ and so

$$\begin{aligned}
T_i^2 v_L &= ((T_i)_{LL}^2 + (q^{-1} + (T_i)_{LL})(q^{-1} + (T_i)_{s_iL,s_iL}))v_L \\
&\qquad + (q^{-1} + (T_i)_{LL})((T_i)_{LL} + (T_i)_{s_iL,s_iL})v_{s_iL} \\
&= (T_i)_{LL}((T_i)_{LL} + (T_i)_{s_iL,s_iL})v_L + q^{-1}(q^{-1} + (T_i)_{LL} + (T_i)_{s_iL,s_iL}))v_L \\
&\qquad + (q^{-1} + (T_i)_{LL})(q - q^{-1})v_{s_iL} \\
&= (T_i)_{LL}(q - q^{-1})v_L + (q^{-1} + (T_i)_{LL})(q - q^{-1})v_{s_iL} + q^{-1}(q^{-1} + q - q^{-1})v_L \\
&= ((q - q^{-1})T_i + 1)v_L.
\end{aligned}$$

The calculations to check the identities (1.1a), (1.1f) and (1.5) are routine. Checking the identity $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$ is more involved. One can proceed as follows. According to the formulas for the action, the operators T_i and T_{i+1} preserve the subspace S spanned by the vectors v_Q indexed by the standard tableaux Q in the set $\{L, s_iL, s_{i+1}L, s_is_{i+1}L, s_{i+1}s_iL, s_is_{i+1}s_iL\}$. Depending on the relative positions of the boxes containing $i, i+1, i+2$ in L, this space is either $1, 2, 3$ or 6 dimensional. Representative cases are when these boxes are positioned in the following ways.

Case (1) Case (2) Case (3) Case (4)

In Case (1) the space S is one dimensional and spanned by the vector v_Q corresponding to the standard tableau

$$\begin{array}{|c|c|c|}\hline a & b & c \\ \hline\end{array}$$

where $a = i$, $b = i+1$, and $c = i+2$. The action of T_i and T_{i+1} on S is given by the matrices

$$\phi_S(T_i) = (\, q\,), \qquad \text{and} \qquad \phi_S(T_{i+1}) = (\, q\,),$$

respectively. In case (2) the space S is two dimensional and spanned by the vectors v_Q corresponding to the standard tableaux

$$\begin{array}{|c|c|}\hline a & b \\ \hline c \\ \cline{1-1}\end{array} \qquad\qquad \begin{array}{|c|c|}\hline a & c \\ \hline b \\ \cline{1-1}\end{array}$$

where $a = i$, $b = i+1$, and $c = i+2$. The action of T_i and T_{i+1} on S is given by the matrices

$$\phi_S(T_i) = \begin{pmatrix} q & 0 \\ 0 & -q^{-1} \end{pmatrix} \qquad \text{and} \qquad \phi_S(T_{i+1}) = \begin{pmatrix} \dfrac{q-q^{-1}}{1-q^4} & \dfrac{q-q^{-5}}{1-q^{-4}} \\ \dfrac{q-q^3}{1-q^4} & \dfrac{q-q^{-1}}{1-q^{-4}} \end{pmatrix}$$

In case (3) the space S is three dimensional and spanned by the vectors v_Q corresponding to the standard tableaux

$$\begin{array}{ccc} & & \boxed{c} \\ \boxed{a} & \boxed{b} & \end{array} \qquad \begin{array}{ccc} & & \boxed{b} \\ \boxed{a} & \boxed{c} & \end{array} \qquad \begin{array}{ccc} & & \boxed{a} \\ \boxed{b} & \boxed{c} & \end{array}$$

where $a = i$, $b = i+1$, and $c = i+2$. The action of T_i and T_{i+1} on S is given by the matrices

$$\phi_S(T_i) = \begin{pmatrix} q & 0 & 0 \\ 0 & \dfrac{q-q^{-1}}{1-q^{2(c_1-c_3)}} & \dfrac{q-q^{2(c_3-c_1)-1}}{1-q^{2(c_3-c_1)}} \\ 0 & \dfrac{q-q^{2(c_1-c_3)-1}}{1-q^{2(c_1-c_3)}} & \dfrac{q-q^{-1}}{1-q^{2(c_3-c_1)}} \end{pmatrix} \qquad \text{and}$$

$$\phi_S(T_{i+1}) = \begin{pmatrix} \dfrac{q-q^{-1}}{1-q^{2(c_2-c_3)}} & \dfrac{q-q^{2(c_3-c_2)-1}}{1-q^{2(c_3-c_2)}} & 0 \\ \dfrac{q-q^{2(c_2-c_3)-1}}{1-q^{2(c_2-c_3)}} & \dfrac{q-q^{-1}}{1-q^{2(c_3-c_2)}} & 0 \\ 0 & 0 & q \end{pmatrix}$$

respectively, where $c_1 = c(L(i))$, $c_2 = c(L(i+1))$ and $c_3 = c(L(i+2))$. In case (4) the space S is six dimensional and spanned by the vectors v_Q corresponding to the standard tableaux

$$\begin{array}{ccc} & & \boxed{c} \\ & \boxed{b} & \\ \boxed{a} & & \end{array} \qquad \begin{array}{ccc} & & \boxed{c} \\ & \boxed{a} & \\ \boxed{b} & & \end{array} \qquad \begin{array}{ccc} & & \boxed{b} \\ & \boxed{c} & \\ \boxed{a} & & \end{array} \qquad \begin{array}{ccc} & & \boxed{a} \\ & \boxed{c} & \\ \boxed{b} & & \end{array} \qquad \begin{array}{ccc} & & \boxed{b} \\ & \boxed{a} & \\ \boxed{c} & & \end{array} \qquad \begin{array}{ccc} & & \boxed{a} \\ & \boxed{b} & \\ \boxed{c} & & \end{array}$$

where $a = i$, $b = i+1$, and $c = i+2$. The action of T_i and T_{i+1} on S is given by the matrices

$$\phi_S(T_i) = \begin{pmatrix} \frac{q-q^{-1}}{1-q^{2d_{12}}} & \frac{q-q^{2d_{21}-1}}{1-q^{2d_{21}}} & 0 & 0 & 0 & 0 \\ \frac{q-q^{2d_{12}-1}}{1-q^{2d_{12}}} & \frac{q-q^{-1}}{1-q^{2d_{21}}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{q-q^{-1}}{1-q^{2d_{13}}} & \frac{q-q^{2d_{31}-1}}{1-q^{2d_{31}}} & 0 & 0 \\ 0 & 0 & \frac{q-q^{2d_{13}-1}}{1-q^{2d_{13}}} & \frac{q-q^{-1}}{1-q^{2d_{31}}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{q-q^{-1}}{1-q^{2d_{23}}} & \frac{q-q^{2d_{32}-1}}{1-q^{2d_{32}}} \\ 0 & 0 & 0 & 0 & \frac{q-q^{2d_{23}-1}}{1-q^{2d_{23}}} & \frac{q-q^{-1}}{1-q^{2d_{32}}} \end{pmatrix}$$

and

$$\phi_S(T_{i+1}) = \begin{pmatrix} \frac{q-q^{-1}}{1-q^{2d_{23}}} & 0 & \frac{q-q^{2d_{32}-1}}{1-q^{2d_{32}}} & 0 & 0 & 0 \\ 0 & \frac{q-q^{-1}}{1-q^{2d_{13}}} & 0 & 0 & \frac{q-q^{2d_{31}-1}}{1-q^{2d_{31}}} & 0 \\ \frac{q-q^{2d_{23}-1}}{1-q^{2d_{23}}} & 0 & \frac{q-q^{-1}}{1-q^{2d_{32}}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{q-q^{-1}}{1-q^{2d_{12}}} & 0 & \frac{q-q^{2d_{21}-1}}{1-q^{2d_{21}}} \\ 0 & \frac{q-q^{2d_{13}-1}}{1-q^{2d_{13}}} & 0 & 0 & \frac{q-q^{-1}}{1-q^{2d_{31}}} & 0 \\ 0 & 0 & 0 & \frac{q-q^{2d_{12}-1}}{1-q^{2d_{12}}} & 0 & \frac{q-q^{-1}}{1-q^{2d_{21}}} \end{pmatrix}$$

where $d_{k\ell} = c(L(i+k-1)) - c(L(i+l-1))$. In each case we compute directly the products $\phi_S(T_i)\phi_S(T_{i+1})\phi_S(T_i)$ and $\phi_S(T_{i+1})\phi_S(T_i)\phi_S(T_{i+1})$ and verify that they are equal. (This proof of the braid relation is, in all essential aspects, the same as that used by Hoefsmit [Ho], Wenzl [Wz] and Ariki and Koike [AK]. For a more elegant but less straightforward proof of this relation see the proof of Theorem 3.1 in [Ra3].) ∎

STEP 2. The module $\tilde{H}^{(c,\lambda/\mu)}$ is irreducible.

Proof. Let L be a standard tableaux of shape λ/μ and define

$$\pi_L = \prod_{i=1}^{n} \prod_{P \neq L} \frac{x_i - q^{2c(P(i))}}{q^{2c(L(i))} - q^{2c(P(i))}},$$

where the second product is over all standard tableaux P of shape λ/μ which are not equal to L. Then π_L is an element of $\tilde{H}_n$ such that

$$\pi_L v_Q = \delta_{LQ} v_L,$$

for all standard tableaux Q of shape λ/μ. This follows from the formula for the action of x_i on $\tilde{H}^{(c,\lambda/\mu)}$ and the fact that the sequence $(q^{2c(L(1))}, \ldots, q^{2c(L(n))})$ completely determines the standard tableau L (Lemma 2.2).

Let N be a nonzero submodule of $\tilde{H}^{(c,\lambda/\mu)}$ and let $v = \sum_Q a_Q v_Q$ be a nonzero element of N. Let L be a standard tableau such that the coefficient a_L is nonzero. Then $\pi_L v = a_L v_L$ and so $v_L \in N$.

By Proposition 2.1 we may identify the set $\mathcal{F}^{\lambda/\mu}$ with an interval in S_n (under Bruhat order). Under this identification the minimal element is the column reading tableau C and there is a chain $C < s_{i_1}C < \cdots < s_{i_p}\cdots s_{i_1}C = L$ such that all elements of the chain are standard tableaux of shape λ/μ. Then, by the definition of the τ_i-operators,

$$\tau_{i_1}\cdots\tau_{i_p} v_L = \kappa v_C,$$

for some constant $\kappa \in \mathbb{C}^*$. It follows that $v_C \in N$.

Let Q be an arbitrary standard tableau of shape λ/μ. Again, there is a chain $C < s_{j_1}C < \cdots < s_{j_p}\cdots s_{j_1}C = Q$ of standard tableaux in $\mathcal{F}^{\lambda/\mu}$ and we have

$$\tau_{j_p}\cdots\tau_{j_1} v_C = \kappa' v_Q,$$

for some $\kappa' \in \mathbb{C}^*$. Thus $v_Q \in N$. It follows that $N = \tilde{H}^{(c,\lambda/\mu)}$. Thus $\tilde{H}^{(c,\lambda/\mu)}$ is irreducible. ∎

STEP 3. The modules $\tilde{H}^{(c,\lambda/\mu)}$ are nonisomorphic.

Proof. Each of the modules $\tilde{H}^{(c,\lambda/\mu)}$ has a unique basis (up to multiplication of each basis vector by a constant) of simultaneous eigenvectors for the x_i. Each basis vector is determined by its weight, the sequence of eigenvalues $(t_1, \ldots, t_n)$ given by

$$x_i v_t = t_i v_t, \qquad \text{for } 1 \le i \le n.$$

By the definition of the action of the x_i, a weight of $\tilde{H}^{(c,\lambda/\mu)}$ is equal to $(q^{2c(L(1))}, \ldots, q^{2c(L(n))})$ for some standard tableau L. By Lemma 2.2, both the standard tableau L and the placed skew shape $(c, \lambda/\mu)$ are determined uniquely by this weight. Thus no two of the modules $\tilde{H}^{(c,\lambda/\mu)}$ can be isomorphic. ∎

STEP 4. If $t = (t_1, \ldots, t_n)$ is the weight of a calibrated $\tilde{H}_n$-module M then $t = (q^{2c(L(1))}, \ldots, q^{2c(L(n))})$ for some standard tableau L of placed skew shape.

Proof. Let m_t be a weight vector in M of weight $t = (t_1, \ldots, t_n)$, i.e.

$$x_i m_t = t_i m_t, \qquad \text{for all } 1 \le i \le n.$$

We want L such that $(q^{2c(L(1))}, \ldots, q^{2c(L(n))}) = (t_1, \ldots, t_n)$. We shall show that if $t_i = t_j$ for $i < j$ then there exist k and ℓ such that $i < k < \ell < j$, $t_k = q^{\pm 2} t_i$ and $t_\ell = q^{\mp 2} t_i$. This will show that

if there are two adjacent boxes of L in the same diagonal then these boxes must be contained in a complete 2×2 block, i.e. if there is a confuguration in L of the form

$$\begin{array}{|c|c}\hline i & \multicolumn{1}{c}{} \\ \hline \multicolumn{1}{c|}{} & j \\ \cline{2-2}\end{array} \quad \text{then } L \text{ must contain} \quad \begin{array}{|c|c|}\hline i & k \\ \hline \ell & j \\ \hline\end{array} \quad \text{or} \quad \begin{array}{|c|c|}\hline i & \ell \\ \hline k & j \\ \hline\end{array}\ .$$

This is sufficient to guarantee that L is of skew shape.

Let $j > i$ be such that $t_j = t_i$ and $j - i$ is minimal. The argument is by induction on the value of $j - i$.

Case 1: $j - i = 1$. Then m_t and $T_i m_t$ are linearly independent. If they were not we would have $T_i m_t = a m_t$ which would give

$$t_i a m_t = x_i T_i m_t = (T_i x_{i+1} - (q - q^{-1}) x_{i+1}) m_t = (a t_{i+1} - (q - q^{-1}) t_{i+1}) m_t = (a - (q - q^{-1})) t_i m_t.$$

Since $q - q^{-1} \neq 0$, this equation implies that $t_i = 0$ which is a contradiction. Now the relations (1.1d) and (1.4) show that

$$\begin{aligned} x_i T_i m_t &= t_i (T_i m_t - (q - q^{-1}) m_t), \\ x_{i+1} T_i m_t &= t_{i+1} (T_i m_t + (q - q^{-1}) m_t), \\ x_j T_i m_t &= t_j T_i m_t, \qquad \text{for all } j \neq i, i+1. \end{aligned}$$

It follows that $T_i m_t$ is an element of M_t^{gen} but not an element of M_t. This is a contradiction to the fact that M is calibrated. So this case is not possible, i.e. t_{i+1} *cannot* equal t_i.

Case 2: $j - i = 2$. Since $t_i \neq t_{i+1}$ and m_t is a weight vector, the vector

$$m_{s_i t} = \left(T_i - \frac{(q - q^{-1}) t_{i+1}}{t_{i+1} - t_i} \right) m_t$$

is a weight vector of weight $t' = s_i t$ (see (3.7a)). Then $t'_i = t'_{i+1}$ and so, by Case 1, $m_{s_i t} = 0$. This implies that

$$T_i m_t = \frac{(q - q^{-1}) t_{i+1}}{t_{i+1} - t_i} m_t.$$

By equation (1.3), all eigenvalues of T_i are either q or $-q^{-1}$. Thus $T_i m_t = \pm q^{\pm 1} m_t$ and so $t_i = q^{\pm 2} t_{i+1}$. A similar argument shows that

$$m_{s_{i+1} t} = \left(T_{i+1} - \frac{(q - q^{-1}) t_{i+2}}{t_{i+2} - t_{i+1}} \right) m_t$$

must be 0 and thus that

$$T_{i+1} m_t = \frac{(q - q^{-1}) t_{i+2}}{t_{i+2} - t_{i+1}} m_t = \frac{(q - q^{-1}) t_i}{t_i - t_{i+1}} m_t = \mp q^{\mp 1} m_t.$$

From $T_i m_t = \pm q^{\pm 1} m_t$ and $T_{i+1} m_t = \mp q^{\mp 1} m_t$ we get

$$\pm q^{\pm 1} m_t = T_i T_{i+1} T_i m_t = T_{i+1} T_i T_{i+1} m_t = \mp q^{\mp 1} m_t.$$

This is impossible since q is not a root of unity. So this case is not possible, i.e. t_{i+2} *cannot* equal t_i.

Induction step. Assume that i and j are such that $t_i = t_j$ and the value $j-i$ is minimal such that this is true.

If $t_{j-1} \neq q^{\pm 2} t_j$ then the vector

$$m_{s_j t} = \left(T_j - \frac{(q-q^{-1})t_j}{t_{j-1} - t_j} \right) m_t$$

is a weight vector of weight $t' = s_j t$ and by (3.7b) this vector is nonzero. Since $t'_i = t_i = t_j = t'_{j-1}$ we can apply the induction hypothesis to conclude that there are k and ℓ with $i < k < \ell < j-1$ such that $t'_k = q^{\pm 2} t'_i$ and $t'_\ell = q^{\mp 2} t'_i$. This implies that $t_k = q^{\pm 2} t_i$ and $t_\ell = q^{\mp 2} t_i$.

Similarly, if $t_i \neq q^{\pm 2} t_{i+1}$ then the vector

$$m_{s_i t} = \left(T_i - \frac{(q-q^{-1})t_{i+1}}{t_{i+1} - t_i} \right) m_t$$

is a weight vector of weight $t' = s_i t$ and by (3.7b) this vector is nonzero. Since $t'_{i+1} = t_i = t_j = t'_j$ we can apply the induction hypothesis to conclude that there are k and ℓ with $i+1 < k < \ell < j$ such that $t'_k = q^{\pm 2} t'_j$ and $t'_\ell = q^{\mp 2} t'_j$. This implies that $t_k = q^{\pm 2} t_i$ and $t_\ell = q^{\mp 2} t_i$.

If we are not in either of the previous cases then $t_{i+1} = q^2 t_i$ or $t_{i+1} = q^{-2} t_i$ and $t_{j-1} = q^2 t_j$ or $t_{j-1} = q^{-2} t_j$. We cannot have $t_{i+1} = t_{j-1}$ since the i and j are such that $j-i$ is minimal such that $t_i = t_j$. Thus $q^{\pm 2} t_{i+1} = q^{\mp 2} t_{j-1} = t_i$. ∎

STEP 5. Suppose that M is an irreducible calibrated $\tilde{H}_n$-module and that m_t is a weight vector in M with weight $t = (t_1, \ldots, t_n)$ such that $t_i = q^{\pm 2} t_{i+1}$. Then $\tau_i m_t = 0$.

Proof. Assume that $m_{s_i t} = \tau_i m_t \neq 0$. Then, by the second identity in Proposition 3.2 (b), $\tau_i m_{s_i t} = 0$. Since M is irreducible there must be some sequence of τ-operators such that

$$\tau_{i_1} \ldots \tau_{i_p} m_{s_i t} = \kappa m_t,$$

with $\kappa \in \mathbb{C}^*$. Assume that $\tau_{i_1} \cdots \tau_{i_p}$ is a minimal length sequence such that this is true. We have $s_{i_1} \cdots s_{i_p} s_i t = t$.

Assume that $s_{i_1} \cdots s_{i_p} s_i \neq 1$. Then there must be $1 \leq i < j \leq n$ such that $t_i = t_j$ (because some nontrivial permutation of the t_i fixes t). Since $s_{i_1} \cdots s_{i_p} s_i$ is of minimal length such that it fixes t it must be a transposition (i,j) for some $i < j$ such that $t_i = t_j$. Furthermore there does not exist $i < k < j$ such that $t_i = t_k$. The element $s_{i_1} \cdots s_{i_p} s_i$ switches the t_i and the t_j in t. In the process of doing this switch by a sequence of simple transpositions there must be some point where t_i and t_j are adjacent and thus there must be some ℓ such that $s_{i_\ell}(s_{i_{\ell+1}} \cdots s_{i_p} s_i t) = s_{i_{\ell+1}} \cdots s_{i_p} s_i t$. Then

$$m_{t'} = \tau_{i_{\ell+1}} \cdots \tau_{i_\ell} \tau_i m_t$$

is a nonzero weight vector in M of weight $t' = s_{i_{\ell+1}} \cdots s_{i_p} s_i t$. Since $s_{i_\ell} t' = t'$ it follows that $t_{i_\ell} = t_{i_\ell + 1}$. Since M is calibrated this is a contradiction to (Case 1 of) STEP 4.

So $s_{i_1} \cdots s_{i_p} s_i = 1$. Let k be minimal such that $s_{i_1} \cdots s_{i_k}$ is not reduced. Assume $p > 1$. Then we can use the braid relations (the third and fourth lines of Proposition 3.2 (b)) to write

$$\kappa m_t = \tau_{j_1} \cdots \tau_{j_{k-2}} \tau_{i_k} \tau_{i_k} \tau_{i_{k+1}} \cdots \tau_{i_p} \tau_i m_t,$$

for some $j_1, \ldots, j_{k-2}$. Then, by the second line of Proposition 3.2 (b),

$$\kappa' m_t = \tau_{j_1} \cdots \tau_{j_{k-2}} \tau_{i_{k+1}} \cdots \tau_{i_p} \tau_i m_t,$$

for some $\kappa' \in \mathbb{C}^*$. This is a contradiction to the minimality of the length of the sequence $\tau_{i_1} \cdots \tau_{i_p}$.

So $p = 1$, $i_p = i$ and $\tau_i \tau_i m_t = \kappa m_t$. This is a contradiction since the second identity in Proposition 3.2 (b) and the assumption that $t_i = q^{\pm 2} t_{i+1}$ imply that $\tau_i \tau_i = 0$. So $\tau_i m_t = 0$. ∎

STEP 6. An irreducible calibrated $\tilde{H}_n$-module M is isomorphic to $\tilde{H}^{(c,\lambda/\mu)}$ for some placed skew shape $(c, \lambda/\mu)$.

Proof. Let m_t be a nonzero weight vector in M. Since M is calibrated STEP 4 implies that there is a placed skew shape $(c, \lambda/\mu)$ and a standard tableau L of shape λ/μ such that $t = (q^{2c(L(1))}, \ldots, q^{2c(L(n))})$. Let us write m_L in place of m_t. Let C be the column reading tableau of shape λ/μ. It follows from Proposition 2.1 that there is a chain $C, s_{i_1}C, \ldots, s_{i_p} \cdots s_{i_1}C = L$ of standard tableaux of shape λ/μ. By (3.7b), all of the τ_{i_j} in this sequence are bijections and so

$$m_C = \tau_{i_1} \cdots \tau_{i_p} m_L$$

is a nonzero weight vector in M. Similarly, if Q is any other standard tableau of shape λ/μ then there is a chain $C, s_{j_1}C, \ldots, s_{j_p} \cdots s_{j_1}C = Q$ and so

$$m_Q = \tau_{j_p} \cdots \tau_{j_1} m_C$$

is a nonzero weight vector in M. Finally, by STEP 5, $\tau_i m_Q = 0$ if $s_i Q$ is not standard (since $q^{2c(Q(i))} = q^{\pm 2} q^{2c(Q(i+1))}$) and so the span of the vectors $\{m_Q \mid Q \text{ a standard tableau of shape } \lambda/\mu\}$ is a submodule of M. Since M is irreducible this must be all of M and the map

$$\begin{array}{ccc} M & \longrightarrow & \tilde{H}^{(c,\lambda/\mu)} \\ \tau_v m_C & \longmapsto & \tau_v v_C \end{array}$$

is an isomorphism of $\tilde{H}_n$-modules. ∎

This completes the proof of Theorem 4.1.

5. "Garnir relations" and an analogue of Young's natural basis

Each of the modules $\tilde{H}^{(c,\lambda/\mu)}$ constructed in Theorem 4.1 has two natural bases:

(a) The "seminormal basis" $\{v_L \mid L \in \mathcal{F}^{\lambda/\mu}\}$ which, up to multiplication of each basis vector by a constant, is given by

$$\tilde{v}_L = \tau_w v_C, \tag{5.1}$$

where C is the column reading tableau of shape λ/μ, w is the permutation such that $L = wC$ and τ_w is as in (3.3).

(b) The "natural basis" $\{n_L \mid L \in \mathcal{F}^{\lambda/\mu}\}$ given by

$$n_L = T_w v_C, \tag{5.2}$$

where C is the column reading tableau of shape λ/μ, w is the permutation such that $L = wC$ and T_w is as defined in (1.6).

Proposition 5.3. *Let $(c, \lambda/\mu)$ be a placed skew shape with n boxes and let $\tilde{H}^{(c,\lambda/\mu)}$ be the $\tilde{H}_n$-module defined in Theorem 4.1. Let C be the column reading tableau of shape λ/μ and let wC denote the tableau C with the entries permuted according to the permutation w. For each standard tableau L let n_L be defined by formula (5.2). Then $\{n_L \mid L \in \mathcal{F}^{\lambda/\mu}\}$ is a basis of $\tilde{H}^{(c,\lambda/\mu)}$.*

Proof. If L is a standard tableau of shape λ/μ let $\tilde{v}_L$ be as given in (5.1). It follows from the formulas defining the module $\tilde{H}^{(c,\lambda/\mu)}$ that the basis $\{\tilde{v}_L \mid L \in \mathcal{F}^{\lambda/\mu}\}$ is simply a renormalized version of the basis $\{v_L \mid L \in \mathcal{F}^{\lambda/\mu}\}$, i.e there are constants $\kappa_L \in \mathbb{C}^*$ such that $\tilde{v}_L = \kappa_L v_L$.

Let $s_{i_1} \cdots s_{i_p} = w$ be a reduced word for w. Then, with notations as in Theorem 4.1,

$$\tilde{v}_L = \tau_{i_1} \ldots \tau_{i_p} v_C = (T_{i_1} - (T_{i_1})_{L_2L_2})(T_{i_2} - (T_{i_1})_{L_3L_3}) \cdots (T_{i_p} - (T_{i_p})_{L_pL_p}) n_C,$$

where $L_j = s_{i_{j+1}} \cdots s_{i_p} C$. Expanding this expression yields

$$\tilde{v}_L = (T_w + \sum_{u<w} b_u T_u) n_C = n_L + \sum_{u<w} b_u n_{uC},$$

for some constants $b_u \in \mathbb{C}$. The second equality is a consequence of the fact that, by Proposition 2.1, the tableaux uC, $u < w$, are always standard. Since $\{\tilde{v}_L \mid L \in \mathcal{F}^{\lambda/\mu}\}$ is a basis it follows from the triangular relation above that $\{n_L \mid L \in \mathcal{F}^{\lambda/\mu}\}$ is also a basis of $\tilde{H}^{(c,\lambda/\mu)}$. ∎

The construction of $\tilde{H}^{(c,\lambda/\mu)}$ in Theorem 4.1 makes the notational assumption that $v_L = 0$, whenever L is not a standard tableau. Formula (5.1) can be used as a definition of $\tilde{v}_L$ even in the case when L is not standard and, from the definition of the action in Theorem 4.1,

$$\tilde{v}_L = 0, \qquad \text{if } L \text{ is not standard.} \tag{5.4}$$

Theorem 5.5 proves that these relations, when expanded in terms of the basis $\{n_L \mid L \in \mathcal{F}^{\lambda/\mu}\}$ are exactly the classical Garnir relations!!

Let λ/μ be a skew shape. A pair of adjacent boxes in the same row of λ/μ determines a *snake* in λ/μ consisting of the boxes in the pair, all the boxes above the righthand box of this pair, and all the boxes below the lefthand box of the pair. See the picture in Theorem 5.5 (b).

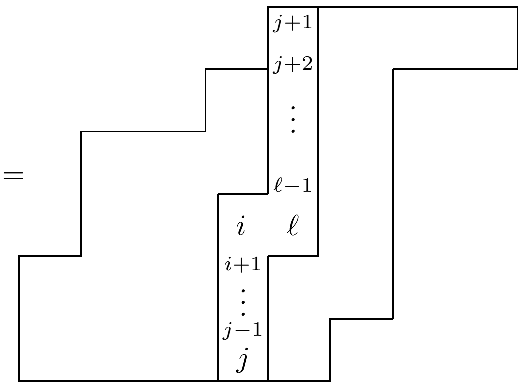
Theorem 5.5. *("Garnir relations") Let $(c, \lambda/\mu)$ be a placed skew shape and let $\tilde{H}^{(c,\lambda/\mu)}$ be the $\tilde{H}_n$-module defined in Theorem 4.1. Let $\{n_L \mid L \in \mathcal{F}^{\lambda/\mu}\}$ be the basis of $\tilde{H}^{(c,\lambda/\mu)}$ defined by Proposition 5.3 and let C be the column reading tableau of shape λ/μ.*

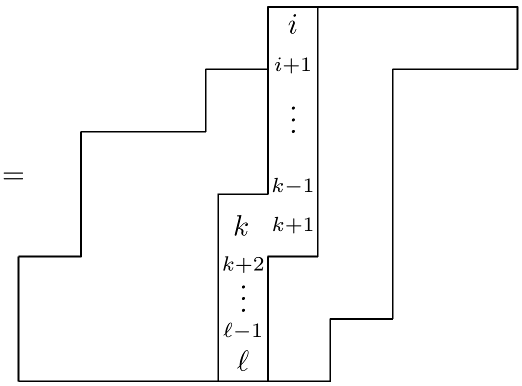
(a) If i and $i+1$ are entries in the same column of C then

$$T_i\, n_C = -q^{-1} n_C.$$

(b) Fix a snake in λ/μ. Let P be the standard tableau which has all entries the same as C except that the entries in the snake are entered in row reading order instead of in column reading

order.

$C =$ 

$P =$ 

Let S_A, S_B and $S_{A\cup B}$ be the subgroups of S_n consisting of the permutations of

$$A = \{i, i+1, \ldots, j\}, \qquad B = \{j+1, \ldots, \ell-1, \ell\}$$

and $A \cup B$, respectively, and let $S_{A\cup B}/(S_A \times S_B)$ be the set of minimal length coset representatives of cosets of $S_A \times S_B$ in $S_{A\cup B}$. The elements of $S_{A\cup B}/(S_A \times S_B)$ are sometimes called the "shuffles" of A and B. Then

$$\begin{aligned} 0 &= \left(\sum_{u \in S_{A\cup B}/(S_A\times S_B)} (-q)^{\ell(w_{A\cup B}) - \ell(w_A w_B) - \ell(u)} T_u \right) n_C \\ &= T_k n_P + \sum_{u \le P} (-q)^{\ell(w_{A\cup B}) - \ell(w_A w_B) - \ell(u)} n_{uC}, \end{aligned}$$

where $\ell(w_{A\cup B}) = \binom{\ell-i+1}{2}$, $\ell(w_A w_B) = \binom{j-i+1}{2}\binom{\ell-j}{2}$ and the last sum is over all standard tableaux uC which are obtained from C by permuting entries which are in the snake.

Proof. Part (a) follows immediately from the definition of $\tilde{H}^{(c,\lambda/\mu)}$ in Theorem 4.1.
(b) The subgroups S_A, S_B and $S_{A\cup B}$ have longest elements

$$w_A = \begin{pmatrix} i & i+1 & \cdots & j-1 & j \\ j & j-1 & \cdots & i+1 & i \end{pmatrix}, \qquad w_B = \begin{pmatrix} j+1 & j+2 & \cdots & \ell-1 & \ell \\ \ell & \ell-1 & \cdots & j+2 & j+1 \end{pmatrix},$$

$$w_{A\cup B} = \begin{pmatrix} i & i+1 & \cdots & \ell-1 & \ell \\ \ell & \ell-1 & \cdots & i+1 & i \end{pmatrix},$$

with lengths $\ell(w_A) = \binom{j-i+1}{2}$, $\ell(w_B) = \binom{\ell-j}{2}$ and $\ell(w_{A\cup B}) = \binom{\ell-i+1}{2}$, respectively, and $w_A w_B$ is the longest element of $S_A \times S_B \subseteq S_{A\cup B}$. Let $t = (t_1, \ldots, t_n) = (a, aq^2, aq^4, \ldots, aq^{2(n-1)})$ where $a = q^{2(c(C(j))-(i-1))} \in \mathbb{C}^*$. The positions of $C(i), \ldots C(\ell)$ in λ/μ are such that

$$\begin{aligned} &(q^{2c(C(i))}, \ldots, q^{2c(C(\ell))}) \\ &\qquad = w_A w_B (q^{2c(C(j))}, q^{2c(C(j-1))}, \ldots, q^{2c(C(i))}, q^{2c(C(\ell))}, q^{2c(C(\ell-1))}, \ldots, q^{2c(C(j+1))}) \\ &\qquad = w_A w_B (q^{2c(C(j))}, q^{2c(C(j))} q^2, \ldots, q^{2c(C(j))} q^{2(\ell-i+1)}) \\ &\qquad = w_A w_B (t_i, \ldots, t_\ell). \end{aligned}$$

By Proposition 3.6 and the fact that $T_s n_C = -q^{-1} n_C$ for all $i \le s \le \ell$, $s \neq j$,

$$\tau_{w_A w_B}\big|_t n_C = \left(\sum_{w \in S_A \times S_B} (-q)^{\ell(w_A w_B) - \ell(w)} T_w\right) n_C = [j-i+1]![\ell - j]! n_C,$$

where $[r]! = [r][r-1]\cdots[2][1]$ and $[p] = (q^p - q^{-p})/(q - q^{-1})$.

Let π be the permutation such that $P = \pi C$ and let $\tilde v_P = \tau_\pi v_C$. Then $s_i \pi w_A w_B = w_{A\cup B}$ and, by (3.5),

$$\begin{aligned}
\tau_i \tilde v_P &= \tau_i \tau_\pi v_C = \tau_i\tau_\pi\big|_{w_A w_B t} n_C \\
&= \frac{1}{[j-i+1]![\ell-j]!} \tau_i \tau_\pi\big|_{w_A w_B t} \tau_{w_A w_B}\big|_t n_C \\
&= \frac{1}{[j-i+1]![\ell-j]!} \tau_i \tau_\pi \tau_{w_A w_B}\big|_t n_C \\
&= \frac{1}{[j-i+1]![\ell-j]!} \tau_{w_{A\cup B}}\big|_t n_C \\
&= \frac{1}{[j-i+1]![\ell-j]!} \left(\sum_{w \in S_{A\cup B}} (-q)^{\ell(w_{A\cup B}) - \ell(w)} T_w\right) n_C.
\end{aligned}$$

Each element $w \in S_{A\cup B}$ has a unique expression $w = uv$ such that $v \in S_A \times S_B$ and $\ell(w) = \ell(u) + \ell(v)$. The left factor u is the minimal length representative of the coset $w(S_A \times S_B)$ in $S_{A\cup B}$. Then

$$\begin{aligned}
&\sum_{w \in S_{A\cup B}} (-q)^{\ell(w_{A\cup B}) - \ell(w)} T_w \\
&\qquad = \left(\sum_{u \in S_{A\cup B}/(S_A\times S_B)} (-q)^{\ell(w_{A\cup B}) - \ell(w_A w_B) - \ell(u)} T_u\right)\left(\sum_{v \in S_A\times S_B} (-q)^{\ell(w_A w_B) - \ell(v)} T_v\right).
\end{aligned}$$

It follows from the formulas for the action on $\tilde H^{(c,\lambda/\mu)}$ that the element $\tilde v_P = \tau_\pi v_C$ is a nonzero multiple of the basis element v_P. Furthermore, by (5.4), $\tau_k \tilde v_P = 0$. So

$$\begin{aligned}
0 &= \tau_i \tilde v_P \\
&= \frac{1}{[j-i+1]![\ell-j]!}\left(\sum_{u \in S_{A\cup B}/(S_A\times S_B)} (-q)^{\ell(w_{A\cup B}) - \ell(w_A w_B) - \ell(u)} T_u\right) \\
&\qquad\qquad \times \left(\sum_{v \in S_A\times S_B} (-q)^{\ell(w_A w_B) - \ell(v)} T_v\right) n_C \\
&= \left(\sum_{u \in S_{A\cup B}/(S_A\times S_B)} (-q)^{\ell(w_{A\cup B}) - \ell(w_A w_B) - \ell(u)} T_u\right) n_C
\end{aligned}$$

For each $u \in S_{A\cup B}/(S_A \times S_B)$ except $u = w_{A\cup B} w_A w_B$, the tableau uC is standard. In fact these are exactly the standard tableaux Q which are obtained by permuting entries of C which are in the snake. The tableau $w_{A\cup B} w_A w_B C = s_k P$. Note that $\ell(w_{A\cup B} w_A w_B) = \ell(w_{A\cup B}) - \ell(w_A w_B)$. Thus we have

$$0 = T_k n_P + \sum_{u \le P} (-q)^{\ell(w_{A\cup B}) - \ell(w_A w_B) - \ell(u)} n_{uC},$$

where the sum over all standard tableaux uC which are obtained from C by permuting entries which are in the snake. ▮

Proposition 5.6. *Let $(c, \lambda/\mu)$ be a placed skew shape and let $\{n_L \mid L$ is a standard tableau of shape $\lambda/\mu\}$ be the basis of the $\tilde{H}_n$-module $\tilde{H}^{(c,\lambda/\mu)}$ which is defined by Proposition 5.3.*

(a) If $w \in S_n$ and L is a standard tableau of shape λ/μ then

$$T_w n_L = \sum_Q b_Q n_Q, \qquad \text{with coefficients } b_Q \in \mathbb{Z}[q, q^{-1}].$$

(b) Assume that the content function c takes values in $\mathbb{Z}$. If $1 \le i \le n$ and L is a standard tableau of shape λ/μ then

$$x_i n_L = \sum_Q b'_Q n_Q, \qquad \text{with coefficients } b'_Q \in \mathbb{Z}[q, q^{-1}].$$

Proof. (a) It is sufficient to show that for all $1 \le d \le n$ and all standard tableaux L of shape λ/μ we have

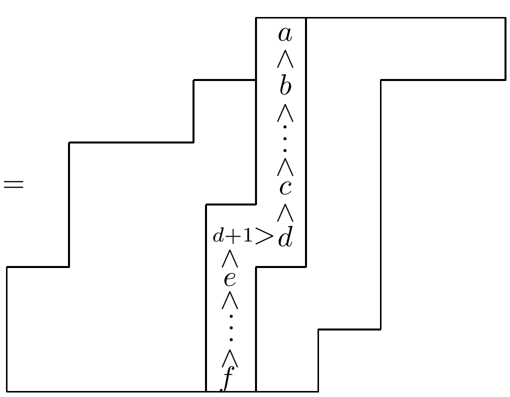
$$T_d n_L = \sum_Q b_Q n_Q,$$

with coefficients $b_Q \in \mathbb{Z}[q, q^{-1}]$. For notational convenience let us identify each standard tableau L of shape λ/μ with the permutation $w \in S_n$ such that $wC = L$, where C is the column reading tableau of shape λ/μ.

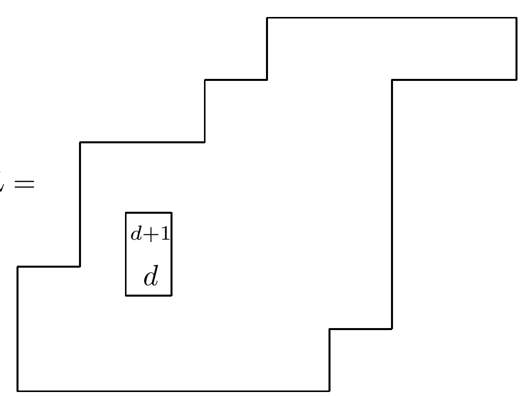
From the definitions

$$\begin{aligned} T_d n_L &= n_{s_d L}, && \text{if } \ell(s_d L) = \ell(L) + 1 \text{ and } s_d L \text{ is standard,} \\ T_d n_L &= (q - q^{-1}) n_L + n_{s_d L}, && \text{if } \ell(s_d L) = \ell(L) - 1. \end{aligned}$$

Let L be a standard tableau such that $\ell(s_d L) = \ell(L) + 1$ and $s_d L$ is not standard. The pair of boxes where the nonstandardness in $s_d L$ occurs in either a row or a column:

$s_dL =$ 

Case (1)

$s_dL =$ 

Case (2)

Case (1): The two boxes in $s_d L$ where the nonstandardness occurs determine a snake in the shape λ/μ consisting of all the boxes above the entry d and all the boxes below the entry $d+1$. Use the same notation as in Theorem 5.5 so that P is the standard tableau which is the same as the

column reading tableau C except that the entries in the snake are in row reading order. Then s_kP is nonstandard and there exists a (unique) $v \in S_n$ such that $s_dL = vs_kP$ and $\ell(s_dL) = \ell(v) + \ell(s_kP)$. The permutation v is given by

$$v = \begin{pmatrix} i & i+1 & \cdots & k-1 & k & k+1 & k+2 & \cdots & \ell \\ a & b & \cdots & c & d & d+1 & e & \cdots & f \end{pmatrix}$$

and $v(x) = L(x)$ for all $x \notin \{i, i+1, \ldots, \ell\}$. Thus

$$T_d n_L = T_d T_L n_C = T_v T_k T_P n_C = T_v T_k n_P.$$

Let $\ell(w_{A\cup B}) = \binom{\ell-i+1}{2}$. By Theorem 5.5

$$T_k n_P = -\sum_{u \le P} (-q)^{\ell(w_{A\cup B}) - \ell(w_A w_B) - \ell(u)} T_u n_C,$$

and so

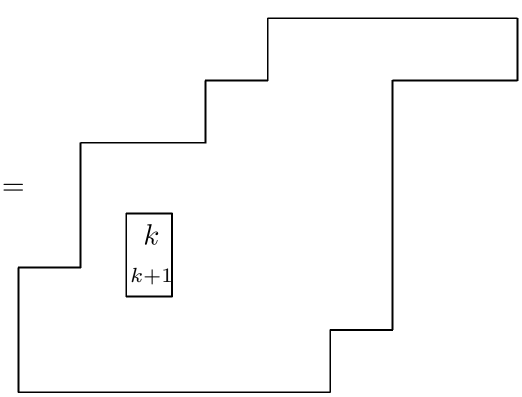
$$T_d n_L = -T_v \left(\sum_{u \le P} (-q)^{\ell(w_{A\cup B}) - \ell(w_A w_B) - \ell(u)} T_u n_C \right).$$

The permutations v and $u \le P$ are such that $\ell(vu) = \ell(v) + \ell(u)$ and so $T_v T_u = T_{vu}$ for all $u \le P$. Furthermore, the tableaux vuC, for $u \le P$, are exactly the standard tableaux which are obtained by permutations of the entries in L which are in the snake. Thus

$$T_d n_L = -\sum_{u \le P} (-q)^{\ell(w_{A\cup B}) - \ell(w_A w_B) - \ell(u)} n_{vuC}.$$

Case (2): Suppose that the boxes containing d and $d+1$ in L are the boxes containing k and $k+1$ in the column reading tableau C.

$C =$



Then $Ls_k = s_dL$ and

$$T_d n_L = T_d T_L n_C = T_L T_k n_C = -q^{-1} T_L n_C = -q^{-1} n_L,$$

by Theorem 5.5 (a).

(b) Let L be a standard tableau of shape λ/μ and let w be the permutation such that $L = wC$, where C is the column reading tableau of shape λ/μ. By repeatedly using the relations (1.4) we obtain

$$x_i n_L = x_i T_w n_C = T_w x_{w^{-1}(i)} n_C + \sum_{v<w} T_v p_v(x_1, \ldots, x_n) n_C,$$

where the $p_v(x_1, \ldots, x_n)$ are polynomials in $x_1, \ldots, x_n$ with coefficients in $\mathbb{Z}[q, q^{-1}]$. If x_i acts on v_C by integral powers of q then we have

$$x_i n_L = T_w q^{2c(C(w^{-1}(i)))} n_C + \sum_{v<w} T_v p_v(q^{2c(C(1))}, \ldots, q^{2c(C(n))}) n_C = \sum_{v \le w} b'_v n_{vC},$$

with coefficients b'_v in $\mathbb{Z}[q, q^{-1}]$. ∎

6. Induction and restriction

Restriction to H_n. Let H_n be the subalgebra of $\tilde{H}_n$ generated by $T_1, \ldots, T_{n-1}$. The elements T_w, $w \in S_n$, form a basis of H_n. Since q is not a root of unity the subalgebra H_n of $\tilde{H}_n$ is semisimple. The irreducible representations of H_n are indexed by the partitions $\nu \vdash n$ and these representations are q-analogues of the irreducible representations of the symmetric group S_n. The following result describes the decomposition of the restriction to H_n of the irreducible $\tilde{H}$-module $\tilde{H}^{(c,\lambda/\mu)}$.

Theorem 6.1. *Let $\tilde{H}^{(c,\lambda/\mu)}$ be the irreducible representation of the affine Hecke algebra $\tilde{H}_n$ which is defined in Theorem 4.1. Then*

$$\tilde{H}^{(c,\lambda/\mu)}\big\downarrow^{\tilde{H}_n}_{H_n} = \sum_{\nu \vdash n} c^\lambda_{\mu\nu} H^\nu,$$

where $c^\lambda_{\mu\nu}$ is the classical Littlewood-Richardson coefficient and H^ν is the irreducible H_n-module indexed by the partition ν.

Proof. If $\beta = (\beta_1, \ldots, \beta_\ell)$ is a composition of n let $\gamma_\beta = \gamma_{\beta_1} \times \cdots \times \gamma_{\beta_\ell} \in S_{\beta_1} \times \cdots \times S_{\beta_\ell}$ where $\gamma_r = (1, 2, \ldots, r) \in S_r$ (in cycle notation). Let $\chi^{(c,\lambda/\mu)}(T_{\gamma_\beta})$ be the trace of the action of the element $T_{\gamma_\beta} \in H_n$ on the $\tilde{H}_n$-module $\tilde{H}^{(c,\lambda/\mu)}$. With notations as in Theorem 4.1

$$\chi^{(c,\lambda/\mu)}(T_{\gamma_\beta}) = \sum_{Q \in \mathcal{F}^{\lambda/\mu}} T_{\gamma_\beta} v_Q \big|_{v_Q},$$

and one can copy (without change) the proof of Theorem 2.20 in [HR2] and obtain

$$\chi^{(c,\lambda/\mu)}(T_{\gamma_\beta}) = \sum_{\mu \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(\ell)} = \lambda} \Delta(\lambda^{(1)}/\mu)\Delta(\lambda^{(2)}/\lambda^{(1)}) \cdots \Delta(\lambda^{(\ell)}/\lambda^{(\ell-1)}),$$

where the sum is over all sequences $\mu \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(\ell)} = \lambda$ such that $|\lambda^{(i)}/\lambda^{(i-1)}| = \beta_i$ and the factor $\Delta(\lambda^{(i)}/\lambda^{(i-1)})$ is given by

$$\Delta(\lambda/\mu) = \begin{cases} (q-q^{-1})^{cc-1} \prod_{bs \in CC} q^{c(bs)-1}(-q^{-1})^{r(bs)-1}, & \text{if } \lambda/\mu \text{ is a border strip,} \\ 0, & \text{otherwise.} \end{cases}$$

In the formula for $\Delta(\lambda/\mu)$: a border strip is a skew shape with at most one box in each diagonal, CC is the set of connected components of λ/μ, cc is the number of connected components of λ/μ, $r(bs)$ is the number of rows of bs, and $c(bs)$ is the number of columns of bs.

Let s_λ denote the Schur function (see [Mac]) and define

$$q_r = \sum_{m=1}^{r} (-q^{-1})^{r-m} q^{m-1} s_{(m1^{r-m})}.$$

By Proposition 6.11(a) in [HR1],

$$q_r s_\mu = \sum_\lambda \Delta(\lambda/\mu) s_\lambda.$$

Letting $q_\beta = q_{\beta_1} \cdots q_{\beta_\ell}$ one can inductively apply this formula to obtain

$$q_\beta s_\mu = \sum_\lambda \chi^{(c,\lambda/\mu)}(T_{\gamma_\beta}) s_\lambda.$$

Thus, with notations as in [Mac] Chapt. I,

$$\begin{aligned}
\chi^{(c,\lambda/\mu)}(T_{\gamma_\beta}) &= \langle q_\beta s_\mu, s_\lambda \rangle \\
&= \langle q_\beta, s_{\lambda/\mu} \rangle, && \text{by [Mac] I (5.1),} \\
&= \sum_\nu c^\lambda_{\mu\nu} \langle q_\beta, s_\nu \rangle, && \text{by [Mac] I (5.3),} \\
&= \sum_\nu c^\lambda_{\mu\nu} \chi^\nu(T_{\gamma_\beta}), && \text{by [Ra1] Theorem 4.14,}
\end{aligned}$$

where $\chi^\nu(T_{\gamma_\beta})$ denotes the irreducible character of H_n evaluated at the element T_{γ_β}. The result follows since, by [Ra1] Theorem 5.1, the characters of H_n are determined by their values on the elements T_{γ_β}. ∎

Classically, the Littlewood-Richardson coefficients describe

(1) The decomposition of the tensor product of two irreducible polynomial representations of $GL_n(\mathbb{C})$, and
(2) The decomposition of an irreducible representation of $S_k \times S_\ell$ when it is induced to $S_{k+\ell}$.

Theorem 6.2 gives an exciting *new* way of interpreting these coefficients. They describe

(3) The decomposition of an irreducible representation $\tilde{H}_n$ when it is restricted to the subalgebra H_n.

Induction from "Young subalgebras". Let k and ℓ be such that $k + \ell = n$. Let

$$\begin{aligned}
\tilde{H}_k &= \text{ the subalgebra of } \tilde{H}_n \text{ generated by } T_i, 1 \le i \le k-1 \text{ and } x_i,\ 1 \le i \le k, \\
\tilde{H}_\ell &= \text{ the subalgebra of } \tilde{H}_n \text{ generated by } T_i,\ k+1 \le i \le n-1, \text{ and } x_i,\ k+1 \le i \le n.
\end{aligned}$$

In this way $\tilde{H}_k \otimes \tilde{H}_\ell$ is naturally a subalgebra of $\tilde{H}_n$.

Let (a, θ) be a placed skew shape with k boxes and let (b, ϕ) be a placed skew shape with ℓ boxes. Number the boxes of θ with $1, \ldots, k$ (as in Section 2, along diagonals from southwest to northeast) and number the boxes of ϕ with $k+1, \ldots, n$, in order to match the imbeddings of $\tilde{H}_k$

and $\tilde{H}_\ell$ in $\tilde{H}_n$. Let $\tilde{H}^{(a,\theta)}$ and $\tilde{H}^{(b,\phi)}$ be the corresponding representations of $\tilde{H}_k$ and $\tilde{H}_\ell$ as defined in Theorem 4.1.

Let $\theta *_v \phi$ (resp. $\theta *_h \phi$) be the skew shape obtained by placing θ and ϕ adjacent to each other in such a way that $\text{box}_{(k+1)}$ of ϕ is immediately above (resp. to the left of) box_k of θ. Let $a \otimes b$ be the content function given by

$$(a \otimes b)(\text{box}_i) = \begin{cases} a(\text{box}_i), & \text{if } 1 \leq i \leq k, \\ b(\text{box}_i), & \text{if } k+1 \leq i \leq \ell, \end{cases}$$

Theorem 6.2. *With notations as above,*

$$\text{Ind}_{\tilde{H}_k \otimes \tilde{H}_\ell}^{\tilde{H}_n}(\tilde{H}^{(a,\theta)} \otimes \tilde{H}^{(b,\phi)}) = \tilde{H}^{(a\otimes b, \theta *_v \phi)} + \tilde{H}^{(a\otimes b, \theta *_h \phi)}$$

in the Grothendieck ring of finite dimensional representations of $\tilde{H}_n$.

Proof. Let $S_n/(S_k \times S_\ell)$ be the set of minimal length coset representatives of the cosets of $S_k \times S_\ell$ in S_n. The module $M = \text{Ind}_{\tilde{H}_k \otimes \tilde{H}_\ell}^{\tilde{H}_n}(\tilde{H}^{(a,\theta)} \otimes \tilde{H}^{(b,\phi)})$ has basis

$$T_w(v_L \otimes v_Q) \qquad \text{where } w \in S_n/(S_k \times S_\ell),\ L \in \mathcal{F}^\theta \text{ and } Q \in \mathcal{F}^\phi.$$

By repeatedly applying the relations (1.4) we obtain

$$x_i(T_w(v_L \otimes v_Q)) = T_w x_{w^{-1}(i)}(v_L \otimes v_Q) + \sum_{u<w} b_u T_u(v_L \otimes v_Q),$$

for some constants $b_u \in \mathbb{C}$. From this we can see that the action of x_i on M is an upper triangular matrix with eigenvalues $q^{2c(P(i))}$ where P runs over the standard tableaux of shapes $\theta *_v \phi$ and $\theta *_h \phi$. It follows that M has

$$\frac{|S_n|\text{Card}(\mathcal{F}^\theta)\text{Card}(\mathcal{F}^\phi)}{|S_k \times S_\ell|}$$

distinct weights. Since this number is exactly the dimension of M, it follows that every generalized weight space of M is one dimensional and thus that M is calibrated. By Theorem 4.1, all irreducible calibrated representations are of the form $\tilde{H}^{(c,\lambda/\mu)}$ for some placed skew shape $(c, \lambda/\mu)$ and by Lemma 2.2 this placed skew shape is completely determined by any one of the weights of the module $\tilde{H}^{(c,\lambda/\mu)}$. Thus, our analysis of the weights of M implies that both $\tilde{H}^{(a\otimes b,\theta *_v \phi)}$ and $\tilde{H}^{(a\otimes b,\theta *_h \phi)}$ are composition factors of M. The result follows since

$$\dim(\tilde{H}^{(a\otimes b,\theta *_v \phi)}) + \dim(\tilde{H}^{(a\otimes b,\theta *_h \phi)}) = \dim(M). \qquad \blacksquare$$

A *ribbon* is a skew shape which has at most one box in each diagonal.

Corollary 6.3. *Let c be the content function given by $c(\text{box}_i) = i - 1$, for $1 \leq i \leq n$. Let $t = (t_1, \ldots, t_n) = (1, q^2, \ldots, q^{2(n-1)})$ and let $\mathbb{C}v_t$ be the one dimensional module for $\mathbb{C}[X] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ given by*

$$x_i v_t = t_i v_t, \qquad \text{for all } 1 \leq i \leq n.$$

In the Grothendieck ring of finite dimensional $\tilde{H}_n$-modules

$$\mathrm{Ind}_{\mathbb{C}[X]}^{\tilde{H}_n}(\mathbb{C}v_t) = \sum_{\lambda/\mu} \tilde{H}^{(c,\lambda/\mu)},$$

where the sum is over all connected ribbons λ/μ with n boxes.

Proof. Since $\mathbb{C}[X] = \tilde{H}_1 \otimes \tilde{H}_1 \otimes \cdots \otimes \tilde{H}_1 \subseteq \tilde{H}_n$ this result can be obtained by repeatedly applying Theorem 6.2. ∎

Theorem 6.2 and Corollary 6.3 are $\tilde{H}_n$-module realizations of the Schur function identities in [Mac] I §5 Ex 21 (a),(b). The module $\mathrm{Ind}_{\mathbb{C}[X]}^{\tilde{H}_n}(\mathbb{C}v_t)$ is a principal series module for $\tilde{H}_n$ (see [Ka]). The identity in Corollary 6.3 describes the composition series of this principal series module. Using the methods of [Ra3] and [Ra4, (1.2) Ex. 2] one can obtain a generalization of this identity (and thus of [Mac] I §5 Ex 21 (b)) which holds for affine Hecke algebras of arbitrary Lie type.

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